Generalized de la Vallée Poussin approximations on $[-1, 1]$ ☆

Woula Themistoclakis[a], Marc Van Barel[b]

[a] C.N.R. National Research Council of Italy, Istituto per le Applicazioni del Calcolo "Mauro Picone", via P. Castellino, 111, 80131 Napoli, Italy.

[b] KU Leuven, Department of Computer Science, KU Leuven, Celestijnenlaan 200A, B-3001 Leuven (Heverlee), Belgium.

Abstract

In this paper a general approach to de la Vallée Poussin means is given and the resulting near best polynomial approximation is stated by developing simple sufficient conditions to guarantee that the Lebesgue constants are uniformly bounded. Not only the continuous case but also the discrete approximation is investigated and a pointwise estimate of the generalized de Vallée Poussin kernel has been stated to this purpose. The theory is illustrated by several numerical experiments.

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1. Introduction

For the construction of stable and efficient projection numerical methods to solve functional equations arising from applications, as well as for several other purposes, one often needs a sequence of polynomial approximation operators, which, on one hand preserve the polynomials up to a certain degree, and on the other hand, they have uniformly bounded operator norm (Lebesgue constants) in the spaces of interest. If in these functional spaces the Weierstrass approximation theorem holds, both the previous properties ensure the convergence to the function we aim to approximate, with the best approximation rate.

In weighted L^p approximation on $[-1, 1]$ with $1 < p < \infty$, under suitable assumptions for the weights, the most prominent example of such approximation operators is the Fourier-Jacobi projection, defined by

$$S_n f(x) := \sum_{j=0}^{n} c_j(f) p_j(x), \qquad c_j(f) := \int_{-1}^{1} f(y) p_j(y) w(y) dy,$$

where $\{p_j(x)\}_j$ are orthonormal Jacobi polynomials associated with the Jacobi weight w.

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Email addresses: woula.themistoclakis@cnr.it (Woula Themistoclakis), marc.vanbarel@cs.kuleuven.be (Marc Van Barel)

Nevertheless, in the limiting cases $p = 1$ or $p = \infty$, it is well-known that neither Fourier nor any other polynomial projection can ensure an optimal approximation, due to the unboundedness of the associated Lebesgue constants, which grow as $\log n$ in the best cases (Faber theorem, 1914).

To overcome this problem, several summation methods can be considered. In particular, in [11] simple sufficient conditions on the weight w have been stated in order that the following polynomial quasi–projection

$$V_n f(x) := \frac{1}{n+1} \sum_{k=n}^{2n} S_k f(x), \tag{1}$$

well-approximates f even for $p = 1, \infty$, exhibiting an arbitrarily fast convergence when f is smooth.

The arithmetic mean (1) generalizes similar means introduced and exploited by the Belgium mathematicians Charles Jean de la Vallée-Poussin in the trigonometric approximation setting. In this context, we recall that using the translation $T_x f(y) := f(x+y)$, the Fourier projection can be defined by the convolution with the Dirichlet kernel D_n, namely

$$\mathcal{S}_n f(x) = (f * D_n)(x) = \frac{1}{2\pi} \int_0^{2\pi} D_n(y) T_x f(y) dy, \qquad D_n(y) = 1 + 2\sum_{k=1}^{n} \cos ky,$$

as well as for all $n > m$, the de la Vallée Poussin (briefly VP) mean

$$\mathcal{V}_n^m f(x) = \frac{1}{2m+1} \sum_{k=n-m}^{n+m} \mathcal{S}_k f(x), \tag{2}$$

can be written as the following convolution product

$$\mathcal{V}_n^m f(x) = (f * v_n^m)(x) = \frac{1}{2\pi} \int_0^{2\pi} v_n^m(y) T_x f(y) dy,$$

where the VP kernel v_n^m is equivalently defined in one of the following two ways

$$v_n^m(y) = \frac{1}{2m+1} \sum_{k=n-m}^{n+m} D_k(x), \qquad \text{or} \qquad v_n^m(y) = \frac{D_n(y) D_m(y)}{D_m(0)}, \tag{3}$$

which lead to the same approximation polynomial by taking into account the trigonometric identity

$$\sum_{k=|n-m|}^{n+m} D_k(x) = D_n(x) D_m(x). \tag{4}$$

In the extension to the algebraic case, where a convolution structure is also given for Jacobi polynomials [8], the Dirichlet kernel is replaced by Darboux kernel, but the identity (4) is no longer satisfied (unless the Chebyshev case) so that the left and right side of (3) lead to different definitions of VP kernels and means. More precisely, if we take the first equality in (3), then we get the algebraic analogous of (2), that is delayed arithmetic means of the Fourier sums of the same kind of (1). On the other hand, the second identity in (3) offers a different perspective, firstly introduced in [2], which leads to the following delayed weighted mean [3]

$$V_n^m f(x) = \sum_{r=n-m}^{n+m} c_{r,n}^m S_r f(x), \qquad c_{r,n}^m = \frac{\int_{-1}^1 K_n(x) K_m(x) K_r(x) (x_0 - x) w(x) dx}{K_m(x_0) \int_{-1}^1 K_r^2(x)(x_0 - x) w(x) dx} \tag{5}$$

where $w(x) = v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ is a given Jacobi weight, $K_n(x) = K_n(x, x_0) = \sum_{j=0}^n p_j(x)p_j(x_0)$ is the associated Darboux kernel and

$$x_0 = \begin{cases} 1 & \text{if } \alpha \geq \beta, \\ -1 & \text{if } \alpha < \beta. \end{cases}$$

The advantage of the VP mean in (5) versus (1), derives from the simplicity of the product defining the VP kernel, that allows you to estimate the Lebesgue constants through the simple application of a Cauchy–Schwartz inequality, obtaining that [3]

$$\|V_n^m f - f\|_\infty \leq \mathcal{C} E_{n-m}(f)_\infty, \qquad E_n(f)_\infty := \inf_{\deg P \leq n} \|f - P\|_\infty \tag{6}$$

holds with $\mathcal{C} > 0$ independent of f, n, m, for all pair of integers $n \sim m$ and all Jacobi weights $w = v^{\alpha,\beta} \in L^1$ such that $\alpha + \beta \geq -1$.
Nevertheless, with respect to the simple arithmetic mean (1), the mean in (5) has nonequal weights $c_{r,n}^m$ to be computed. In general this is not an easy task from a computational point of view, but a recurrence relation can be applied to this purpose [2, 5].

Further generalizations of VP means have been more recently considered by Sloan and Womersley in [13, 14], where delayed VP means have been constructed from suitable filter functions. They proved the near best behaviour displayed by (6) on the surface of the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ for any dimension $d \geq 1$, basing their proof on the crucial property that the kernel has a nonnegative sign.

Inspired by their work, in this paper we deal with the following generalized VP (briefly GVP) mean

$$V^{N,M} f(x) = \sum_{k=N}^{M} d_k^{N,M} S_k f(x), \qquad \text{with} \qquad \sum_{k=N}^{M} d_k^{N,M} = 1,$$

where $d_k^{N,M}$ are arbitrary weights, which include all the known cases from literature.

We wonder what are the conditions (on the weight coefficients and on the involved Jacobi weights) sufficient to get uniformly bounded Lebesgue constants. Our analysis covers all the possible generalizations of VP means, including the cases when no information on the signs is available. For all $1 \leq p \leq \infty$, under simple assumptions on the exponents of the Jacobi weights $w = v^{\alpha,\beta}$ and $u = v^{\gamma,\delta}$, for all $N \sim M$ we provide the next optimal error bound for weighted approximation

$$E_M(f)_{u,p} \leq \|(V^{N,M} f - f)u\|_p \leq \mathcal{C} E_N(f)_{u,p}, \qquad \mathcal{C} \neq \mathcal{C}(N, M, f), \tag{7}$$

where $E_N(f)_{u,p} = \inf_{\deg P \leq N} \|(f - P)u\|_p$, and throughout this paper, $\mathcal{C}$ denotes a positive constant which can take different values in different formulas, and we write $\mathcal{C} \neq \mathcal{C}(N, M, f, ..)$ in the case $\mathcal{C}$ is independent of $N, M, f, ..$

Similarly to [13], setting $d_{M+1}^{N,M} = 0$, we suppose that the weight coefficients satisfy

$$\sum_{k=N}^{M} |d_{k+1}^{N,M} - d_k^{N,M}| \leq \frac{\mathcal{C}}{N}, \qquad \mathcal{C} \neq \mathcal{C}(N, M), \tag{8}$$

but they do not necessarily come from a unique filter function and can have arbitrary signs. Based on (8) and on a pointwise estimate of delayed sums of Darboux kernels obtained from an idea in [10], we state a pointwise estimate of the GVP kernel, which we use in proving (7) in order to overcome the difficulty of a nonconstant sign.

Generalizing previous results stated in [16], we also investigate the discrete GVP approximation, obtained by discretizing the Fourier coefficients via Gauss–Jacobi quadrature.

The theory is completed by several numerical experiments which further exploit both the continuous and discrete GVP mean approximation, showing that the theoretical assumptions on the parameters $\alpha, \beta, \gamma, \delta$ are sufficient but not necessary, and besides the decay required in (8), the intrinsic nature of the filter strongly influence the final result.

The paper is divided in the following sections. Section 2 introduces the continuous as well as the discrete GVP mean based on arbitrary filter coefficients. In Section 3, the weighted approximation provided by the continuous GVP operator is studied. In Section 4, this is done for the discrete case. For improving the readability of the paper, the technicalities of the proofs of both the continuous and discrete cases can be found in the dedicated Section 5. Section 6 illustrates the theory of the uniform boundedness of the Lebesgue constants by some numerical experiments and shows that the GVP mean decreases the Gibbs phenomenon. Finally, Section 7 gives the conclusions.

2. Generalized VP means

Let $w(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ be a Jacobi weight with exponents $\alpha, \beta > -1$, and denote by $p_n(x)$ the associated orthonormal Jacobi polynomial of degree n and positive leading coefficient, satisfying

$$\int_{-1}^{1} p_n(x)p_m(x)w(x)dx = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{9}$$

For any pair of positive integers $N < M$ let be also given a uniformly bounded sequence of filter coefficients $h_j^{N,M}$ satisfying

$$h_j^{N,M} = \begin{cases} 1 & \text{if } j \leq N, \\ 0 & \text{if } j > M, \end{cases} \qquad j = 0, 1, 2, \dots \tag{10}$$

Usually in literature (see e.g. [6, 13, 14]), these filter coefficients derive from a sufficiently smooth filter function h , by setting

$$h_j^{N,M} = h\left(1 + \frac{j-N}{M+1}\right), \qquad j = 0, 1, \dots, \tag{11}$$

where the support of h is supposed to be $[0, 2]$ with $h(x) = 1$, for all $x \in [0, 1]$.
Nevertheless in our general approach, the filter coefficients are not necessarily connected to a function, but they are required to be arbitrary real numbers satisfying (10) and uniformly bounded w.r.t. N, M.

Under this setting, we define the *GVP kernel* as follows

$$v^{N,M}(x, y) := \sum_{j=0}^{\infty} h_j^{N,M} p_j(x)p_j(y), \qquad x, y \in [-1, 1]. \tag{12}$$

By means of such kernel, the *GVP operator* is defined in the following standard way

$$V^{N,M}f(x) := \int_{-1}^{1} f(y)v^{N,M}(x, y)w(y)dy, \tag{13}$$

and for any fixed number $n \in \mathbb{N}$ of Jacobi abscissas, the *discrete GVP operator* is given by

$$\tilde{V}_n^{N,M}f(x) := \sum_{i=1}^{n} \lambda_i f(x_i)v^{N,M}(x, x_i), \tag{14}$$

where $\lambda_i = [\sum_{j=0}^{n-1} p_j^2(x_i)]^{-1}$ are the Christoffel numbers, being x_i the zeros of $p_n(x)$.

We observe that $\tilde{V}_n^{N,M} f$ can be deduced from $V^{N,M} f$ by applying the following Gauss-Jacobi quadrature rule

$$\int_{-1}^{1} P(x)w(x)dx = \sum_{i=1}^{n} \lambda_i P(x_i), \qquad \deg P \le 2n-1. \tag{15}$$

Moreover, we note that, by (10), the summation in (12) is finite and $v^{N,M}(x,y)$ is a polynomial of degree at most M, w.r.t. both the variables x and y. Consequently, by (13) and (14), both the continuous and discrete GVP means of an arbitrary function f, are polynomials of degree at most M, which can be equivalently written as follows

$$V^{N,M} f(x) \;=\; \sum_{j=0}^{M} h_j^{N,M} c_j(f) p_j(x), \qquad c_j(f) := \int_{-1}^{1} f(y)p_j(y)w(y)dy, \tag{16}$$

$$\tilde{V}_n^{N,M} f(x) \;=\; \sum_{j=0}^{M} h_j^{N,M} \tilde{c}_{n,j}(f) p_j(x), \qquad \tilde{c}_{n,j}(f) := \sum_{k=1}^{n} \lambda_k f(x_k) p_j(x_k). \tag{17}$$

In the limiting case $N = M$, these polynomials reduce to the classical Fourier projection and its discrete counterpart (that is the Lagrange operator when $n = N+1$), namely

$$V^{N,N} f(x) \;=\; S_N f(x) := \sum_{j=0}^{N} c_j(f) p_j(x), \tag{18}$$

$$\tilde{V}_n^{N,N} f(x) \;=\; \tilde{S}_{N,n} f(x) := \sum_{j=0}^{N} \tilde{c}_{n,j}(f) p_j(x). \tag{19}$$

But in our setting we supposed $0 < N < M$, so that the GVP means result to be delayed weighted means of the previous Fourier projections. More precisely, by applying the next summation by part formula

$$\sum_{j=N}^{M} a_j b_j = -\sum_{j=N}^{M-1} s_j \Delta a_j + a_M s_M, \qquad s_j := \sum_{r=N}^{j} b_r, \quad \Delta a_j := a_{j+1} - a_j \tag{20}$$

to the sums in (16) or (17), we get

$$V^{N,M} f(x) = \sum_{j=N}^{M} d_j^{N,M} S_j f(x), \qquad \tilde{V}_n^{N,M} f(x) = \sum_{j=N}^{M} d_j^{N,M} \tilde{S}_{j,n} f(x), \tag{21}$$

where $d_j^{N,M} := -\Delta h_j^{N,M} = h_j^{N,M} - h_{j+1}^{N,M}$. Moreover, recalling (10), we have

$$\sum_{j=N}^{M} d_j^{N,M} = h_N^{N,M} - h_{M+1}^{N,M} = 1.$$

Consequently, denoted by $\mathbb{P}_N$ the set of all polynomials of degree at most N, we get

$$V^{N,M} P = \sum_{j=N}^{M} d_j^{N,M} S_j P = \sum_{j=N}^{M} d_j^{N,M} P = P, \qquad \forall P \in \mathbb{P}_N.$$

Hence, the GVP operator $V^{N,M} : f \to V^{N,M} f$ is a polynomial quasi–projection, which maps any function into $\mathbb{P}_M$ and reduces to the identity on the set $\mathbb{P}_N$, being

$$V^{N,M} P = P, \qquad \forall P \in \mathbb{P}_N. \tag{22}$$

The next proposition states the same invariance property for the discrete GVP means.

Proposition 2.1. *For all $n \in \mathbb{N}$ and any pair of positive integers $N < M$ satisfying $2n > (N + M)$, we have*

$$\tilde{V}_n^{N,M} P = P, \qquad \forall P \in \mathbb{P}_N. \tag{23}$$

Proof. We observe that for all $P \in \mathbb{P}_N$, the polynomial $Q(y) := P(y)v^{N,M}(x, y)$ has degree at most $N+M$, which is supposed to be less than or equal to the degree $2n-1$ of exactness of (15). Consequently, by (13)–(15), we get

$$\tilde{V}_n^{N,M} P = \sum_{i=1}^{n} \lambda_i Q(x_i) = \int_{-1}^{1} Q(y)w(y)dy = V^{N,M} P = P, \qquad \forall P \in \mathbb{P}_N.$$

$\diamondsuit$

We remark that if $2n \le (N + M)$ then (23) doesn't hold, but in the case $2n \ge (M + 1)$ we have the invariance on polynomials of degree at most $2n - (M + 1)$, i.e.

$$\tilde{V}_n^{N,M} P = P, \qquad \forall P \in \mathbb{P}_{2n-(M+1)}. \tag{24}$$

3. Approximation provided by continuous GVP means

In this section, for a given Jacobi weight $u = v^{\gamma,\delta}$, we aim to study the approximation provided by continuous GVP means in the space $L^p_u := \{f : \|fu\|_p < \infty\}$, $1 \le p \le \infty$, by estimating the error $\|(V^{N,M} f - f)u\|_p$.

In order to measure the approximation degree, we are going to make a comparison with the error of best polynomial approximation in L^p_u, namely

$$E_n(f)_{u,p} := \inf_{P \in \mathbb{P}_n} \|(f - P)u\|_p, \qquad f \in L^p_u, \quad 1 \le p \le \infty, \quad n \in \mathbb{N}.$$

Since $V^{N,M} f \in \mathbb{P}_M$ for all $f \in L^p_u$, then we certainly have

$$\|(V^{N,M} f - f)u\|_p \ge E_M(f)_{u,p}, \qquad \forall N < M. \tag{25}$$

On the other hand, by virtue of (22) it is easy to realize that the next properties are equivalent:

$$\begin{array}{lll} (i) & \|(V^{N,M} f - f)u\|_p \le \mathcal{C} E_N(f)_{u,p} & \text{holds } \forall f \in L^p_u \text{ with } \mathcal{C} \ne \mathcal{C}(N, M, f), \\ (ii) & \|(V^{N,M} f)u\|_p \le \mathcal{C}\|fu\|_p & \text{holds } \forall f \in L^p_u \text{ with } \mathcal{C} \ne \mathcal{C}(N, M, f). \end{array}$$

Hence, the error analysis reduces to investigate whether the so–called Lebesgue constants

$$\|V^{N,M}\|_{u,p} := \|V^{N,M}\|_{L^p_u \to L^p_u} = \sup_{f \ne 0} \frac{\|(V^{N,M} f)u\|_p}{\|fu\|_p}$$

are uniformly bounded w.r.t. N, M. This is stated in the next theorem for all pair of positive integers $N < M$ such that $N \sim M$, where by this notation we mean that $\mathcal{C}^{-1} M \le N \le \mathcal{C} M$ holds with $\mathcal{C} \ne \mathcal{C}(N, M)$.

Theorem 3.1. *Let $1 \le p \le \infty$ and assume that $w = v^{\alpha,\beta}$ and $u = v^{\gamma,\delta}$ are such that*

$$-1 < \gamma - \delta - \frac{\alpha - \beta}{2} < 1, \tag{26}$$

and satisfy the following conditions

$$\begin{cases} \frac{\alpha}{2} - \frac{1}{4} < \gamma + \frac{1}{p} \le \frac{\alpha}{2} + \frac{5}{4}, & \text{and} \quad 0 < \gamma + \frac{1}{p} < \alpha + 1, \\ \frac{\beta}{2} - \frac{1}{4} < \delta + \frac{1}{p} \le \frac{\beta}{2} + \frac{5}{4}, & \text{and} \quad 0 < \delta + \frac{1}{p} < \beta + 1, \end{cases} \tag{27}$$

where for $p = \infty$, we intend $\frac{1}{p} = 0$ and the cases $\gamma = 0$ and $\delta = 0$ are also included. Moreover, suppose that the continuous GVP means associated with w (cf. (13)) are defined by uniformly bounded filter coefficients $h_j^{N,M}$ satisfying (10) and

$$\sum_{j=N}^{M} \left| \Delta^2 h_j^{N,M} \right| \le \frac{\mathcal{C}}{N}, \qquad \mathcal{C} \ne \mathcal{C}(N, M). \tag{28}$$

Then for all pair of integers $N < M$ with $N \sim M$ and for any $f \in L^p_u$, we have

$$\|(V^{N,M} f)u\|_p \le \mathcal{C} \|fu\|_p, \qquad \mathcal{C} \ne \mathcal{C}(N, M, f). \tag{29}$$

Proof. We are going to examine only the cases $p = \infty$ and $p = 1$, since from these ones the case $1 < p < \infty$ follows by applying the following interpolation theorem [7, Corollary 2.2]

INTERPOLATION THEOREM. Let $u_1 \in L^1$ and $u_0 \in L^\infty$ be two arbitrary weight functions. If T is a linear operator such that the maps $T : L^1_{u_1} \to L^1_{u_1}$ and $T : L^\infty_{u_0} \to L^\infty_{u_0}$ are continuous, then also

$$T : L^p_u \to L^p_u, \qquad u = u_1^{\frac{1}{p}} u_0^{1-\frac{1}{p}}, \qquad 1 < p < \infty,$$

is a continuous map. Moreover we have $\|T\|_{L^p_u \to L^p_u} \le \mathcal{C} \max\{\|T\|_{L^1_{u_1} \to L^1_{u_1}}, \|T\|_{L^\infty_{u_0} \to L^\infty_{u_0}}\}$.

<u>*Case $p = \infty$.*</u> Firstly we recall that for any polynomial $P \in \mathbb{P}_n$, for all $1 \le p \le \infty$ and for all Jacobi weight $v \in L^p$, the following Remez inequality holds (see e.g. [1, p.91 (B)])

$$\|Pv\|_{L^p[-1,1]} \le \mathcal{C} \|Pv\|_{L^p[-1+\mathcal{C}n^{-2}, 1-\mathcal{C}n^{-2}]}, \qquad \mathcal{C} \ne \mathcal{C}(n, P). \tag{30}$$

Hence, applying (30) with $p = \infty$ and $P = V^{N,M} f$, for any $f \in L^\infty_u$, we have

$$\begin{aligned} \|(V^{N,M} f)u\|_\infty &\le \mathcal{C} \sup_{|x| \le 1 - \frac{\mathcal{C}}{M^2}} u(x) \left| \int_{-1}^{1} v^{N,M}(x,y) f(y) w(y) dy \right| \\ &\le \mathcal{C} \sup_{|x| \le 1 - \frac{\mathcal{C}}{M^2}} u(x) \int_{-1}^{1} |v^{N,M}(x,y)| |f(y)| w(y) dy \\ &\le \mathcal{C} \|fu\|_\infty \sup_{|x| \le 1 - \frac{\mathcal{C}}{M^2}} u(x) \int_{-1}^{1} |v^{N,M}(x,y)| \frac{w(y)}{u(y)} dy, \end{aligned}$$

and we get the statement once proved that

$$\sup_{|x| \le 1 - \frac{\mathcal{C}}{M^2}} \left[u(x) \int_{-1}^{1} |v^{N,M}(x,y)| \frac{w(y)}{u(y)} dy \right] \le \mathcal{C} \ne \mathcal{C}(N, M). \tag{31}$$

The proof of this result is rather technical and it has been shifted in the Appendix (cf. Lemma 5.3) for a better readability of the paper.

Case $p = 1$. In this dual case, it is easy to check that

$$\|V^{N,M}\|_{u,1} \leq \mathcal{C} \sup_{|y|\leq 1-\frac{\mathcal{C}}{M^2}} \left[\frac{w(y)}{u(y)} \int_{-1}^{1} |v^{N,M}(x,y)| u(x) dx\right],$$

and again the statement follows from Lemma 5.3, since condition (27) with $p = 1$, assures that (54) holds by replacing γ (resp. δ) with $\alpha - \gamma$ (resp. $\beta - \delta$). $\diamondsuit$

Remark 3.2. *We remark that in Theorem 3.1 the hypotheses (26)–(27) on the weights are surely satisfied if $u \in L^p$ and $w \in L^1$ are such that $wu^{-1} \in L^1$ and they satisfy*

$$\frac{u\varphi^\nu}{\sqrt{w\varphi}} \in L^p \qquad \text{and} \qquad \frac{\sqrt{w\varphi}}{u\varphi^\nu} \in L^q \qquad\qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right) \tag{32}$$

for some $\nu \in [0, 1]$. Nevertheless, we note that the previous assumptions are sufficient, but not necessary for (26)–(27).

Corollary 3.3. *Under the assumptions of Theorem 3.1, for all $f \in L^p_u$, we have*

$$E_M(f)_{u,p} \leq \|(V^{N,M} f - f)u\|_p \leq \mathcal{C} E_N(f)_{u,p}, \qquad \mathcal{C} \neq \mathcal{C}(N, M, f). \tag{33}$$

4. Approximation provided by discrete GVP means

In the discrete case, for brevity, we deal only with the most significant case of the uniform norm and state the next result.

Theorem 4.1. *Let $w = v^{\alpha,\beta}$ $(\alpha, \beta > -1)$ and $u = v^{\gamma,\delta}$ $(\gamma, \delta \geq 0)$ be such that the following conditions hold simultaneously*

$$-1 < \gamma - \delta - \frac{\alpha - \beta}{2} < 1, \tag{34}$$

$$\frac{\alpha}{2} - \frac{1}{4} < \gamma \leq \frac{\alpha}{2} + \frac{5}{4}, \tag{35}$$

$$\frac{\beta}{2} - \frac{1}{4} < \delta \leq \frac{\beta}{2} + \frac{5}{4}, \tag{36}$$

Moreover, let $f \in L^\infty_u$ be everywhere defined on $]-1, 1[$ and suppose that the discrete GVP means $\tilde{V}_n^{N,M} f$ associated with w are defined by uniformly bounded filter coefficients $h_j^{N,M}$ satisfying (10) and (28).
Then for all positive integers $n \sim N \sim M$ such that $N < M$ and $2n \geq M + 1$, we have

$$E_M(f)_{u,\infty} \leq \|(\tilde{V}_n^{N,M} f - f)u\|_\infty \leq \mathcal{C} E_{2n-M-1}(f)_{u,\infty}, \qquad \mathcal{C} \neq \mathcal{C}(N, M, n, f). \tag{37}$$

Proof. The left–hand side inequality follows from the fact that $\tilde{V}_n^{N,M} f \in \mathbb{P}_M$.
In order to prove the right–hand side inequality in (37), we suppose that the Jacobi zeros x_k, $k = 1, \ldots, n$, are ordered as follows

$$-1 =: x_0 < x_1 < \ldots < x_n < x_{n+1} := 1. \tag{38}$$

Then, set $\Delta x_k = x_{k+1} - x_k$ and recalling that [12]

$$\lambda_k \le \mathcal{C} w(x_k)\Delta x_k, \qquad k = 1, \ldots, n, \qquad \mathcal{C} \ne \mathcal{C}(n,k),$$

by Remez inequality (30), we have

$$\begin{aligned}
\|(\tilde{V}_n^{N,M} f)u\|_\infty &\le \mathcal{C} \sup_{|x|\le 1-\frac{\mathcal{C}}{M^2}} \left| u(x) \sum_{k=1}^n \lambda_k f(x_k) v^{N,M}(x,x_k) \right| \\
&\le \mathcal{C} \sup_{|x|\le 1-\frac{\mathcal{C}}{M^2}} u(x) \sum_{k=1}^n \lambda_k \left| f(x_k) v^{N,M}(x,x_k) \right| \\
&\le \mathcal{C} \max_{1\le k\le n} |f(x_k)u(x_k)| \sup_{|x|\le 1-\frac{\mathcal{C}}{M^2}} u(x) \sum_{k=1}^n \lambda_k \frac{|v^{N,M}(x,x_k)|}{u(x_k)} \\
&\le \mathcal{C}\|fu\|_\infty \sup_{|x|\le 1-\frac{\mathcal{C}}{M^2}} u(x) \sum_{k=1}^n \frac{w(x_k)}{u(x_k)} |v^{N,M}(x,x_k)| \Delta x_k.
\end{aligned}$$

On the other hand, in the Appendix (cf. Lemma 5.4) it has been proved that

$$\sup_{|x|\le 1-\frac{\mathcal{C}}{M^2}} \left(u(x) \sum_{k=1}^n \frac{w(x_k)}{u(x_k)} |v^{N,M}(x,x_k)| \Delta x_k \right) \le \mathcal{C} \ne \mathcal{C}(n,N,M), \tag{39}$$

so that we obtain

$$\|(\tilde{V}_n^{N,M} f)u\|_\infty \le \mathcal{C}\|fu\|_\infty, \qquad \mathcal{C} \ne \mathcal{C}(n,N,M,f). \tag{40}$$

Consequently, if $P^*_{2n-M-1} \in \mathbb{P}_{2n-M-1}$ is an optimal polynomial for the weighted uniform approximation of f, namely if $\|(f - P^*_{2n-M-1})u\|_\infty \le \mathcal{C}E_{2n-M-1}(f)_{u,\infty}$ holds with $\mathcal{C} \ne \mathcal{C}(n,M,f)$, then by (24) and (40) we have

$$\|(\tilde{V}_n^{N,M} f - f)u\|_\infty \le \|\tilde{V}_n^{N,M}(f - P^*_{2n-M-1})u\|_\infty + \|(f - P^*_{2n-M-1})u\|_\infty \le \mathcal{C}E_{2n-M-1}(f)_{u,\infty}$$

and the statement follows. $\diamondsuit$

By virtue of the previous theorem, the discrete GVP means $\tilde{V}_n^{N,M}$ share the near best behaviour of the approximation we have seen for the continuous GVP means $V^{N,M}$, but in the discrete case we have one more parameter n to deal with. In Section 6 the choice of this parameter has been further investigated observing the behaviour of the Lebesgue constants

$$\|\tilde{V}_n^{N,M}\|_{u,\infty} = \sup_{f\ne 0} \frac{\|(\tilde{V}_n^{N,M} f)u\|_\infty}{\|fu\|_\infty} \sim \sup_{|x|\le 1} \left(u(x) \sum_{k=1}^n \frac{w(x_k)}{u(x_k)} |v^{N,M}(x,x_k)| \Delta x_k \right).$$

5. Appendix to Sections 3 and 4: technicalities

This section is devoted to the proof of the inequalities (31) and (39), on which we based the proof of Theorems 3.1 and 4.1. To this aim we first state a pointwise estimate of the GVP kernel.

Throughout this section, $\mathcal{C}$ denotes a positive constant (taking different values in different formulas) such that $\mathcal{C} \ne \mathcal{C}(N,M,x,y)$ and we write $A \sim B$ to mean $\mathcal{C}^{-1}A \le B \le \mathcal{C}A$. Moreover, in order to avoid confusion between the different considered weights, we make explicit the underlying weight function w by using the notations $p_n(w,x) = p_n(x)$ and $v^{N,M}(w,x,y) = v^{N,M}(x,y)$.

Firstly, let us derive some equivalent expressions of the generalized VP kernel

$$v^{N,M}(w,x,y) = \sum_{j=0}^{M} h_j^{N,M} p_j(w,x)p_j(w,y). \tag{41}$$

Note that if we apply two times the summation by part formula (20) to the sum (41), then in terms of the Darboux kernels $K_n(w,x,y) := \sum_{r=0}^{n} p_r(w,x)p_r(w,y)$, we get

$$v^{N,M}(w,x,y) = \sum_{j=N}^{M} d_j^{N,M} K_j(w,x,y), \qquad d_j^{N,M} = h_j^{N,M} - h_{j+1}^{N,M} \tag{42}$$

$$v^{N,M}(w,x,y) = \sum_{j=N}^{M} D_j^{N,M} \left[\sum_{r=N}^{j} K_r(w,x,y)\right], \qquad D_j^{N,M} = -\Delta d_j^{N,M} = \Delta^2 h_j^{N,M}, \tag{43}$$

where we recall that $\Delta^2 h_j = \Delta\Delta h_j = h_{j+2} - 2h_{j+1} + h_j$.

Starting from (43) and applying well-known properties of the Jacobi polynomials, in the sequel we are going to pointwise estimate the GVP kernel.

To this aim, we note that by (43) we get

$$|v^{N,M}(w,x,y)| \le \left(\sum_{j=N}^{M} \left|D_j^{N,M}\right|\right) \sup_{N\le j\le M} \left|\sum_{r=N}^{j} K_r(w,x,y)\right|. \tag{44}$$

The next lemma provides a pointwise estimate of delayed sums of Darboux kernels

Lemma 5.1. *Let w be a given Jacobi weight. Then for all pairs of positive integers $N < M$ and for all $x \neq y \in \left[-1 + \frac{\mathcal{C}}{N^2}, 1 - \frac{\mathcal{C}}{N^2}\right]$, we have*

$$\left|\sum_{r=N}^{M} K_r(w,x,y)\right| \le \mathcal{C}\, \frac{E_N^{\pm}(x,y)}{|x-y|\sqrt{w(x)\varphi(x)}\sqrt{w(y)\varphi(y)}}, \qquad \mathcal{C} \neq \mathcal{C}(N,M,x,y) \tag{45}$$

where we set $\varphi(x) := \sqrt{1-x^2}$ and

$$E_N^{\pm}(x,y) := \frac{(\sqrt{1\pm x} + \sqrt{1\pm y})^2}{|x-y|} + \frac{\sqrt{1\pm x} + \sqrt{1\pm y}}{N\sqrt{1-x^2}\,\sqrt{1-y^2}}. \tag{46}$$

Proof of Lemma 5.1. In order to state (45), we set $\overline{w}(x) = w(x)(1-x)$ and recall that (see e.g. [15, pag.71])

$$K_m(w,x,y) = b_m \frac{p_m(\overline{w},x)p_m(w,y)(1-x) - p_m(w,x)p_m(\overline{w},y)(1-y)}{y-x} \tag{47}$$

holds for all $m \in \mathbb{N}$, being $b_m := \sqrt{\frac{2(m+\alpha+\beta+1)(m+\alpha+1)}{(2m+\alpha+\beta+1)(2m+\alpha+\beta+2)}}$ for $w = v^{\alpha,\beta}$.

Then we apply to $p_m(w)$ and $p_m(\overline{w})$ in (47), the next asymptotic formula (see e.g. [15, Theorem 8.21.13])

$$p_m(v^{\alpha,\beta}, \cos t) = \frac{a_m}{\sqrt{v^{\alpha,\beta}(\cos t)\varphi(\cos t)}} \left[\cos(mt+\nu) + \frac{\mathcal{O}(1)}{m \sin t}\right], \quad \frac{\mathcal{C}}{m} \le t \le \pi - \frac{\mathcal{C}}{m}, \tag{48}$$

where $\nu := \frac{(2\alpha+2\beta+1)t}{2} - \frac{\pi}{2}(\alpha + \frac{1}{2})$, and $a_m = \sqrt{\frac{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}{\pi m\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}}$.

Hence, set $x = \cos t,\ y = \cos s$, by (47) and (48) we get

$$\begin{aligned}\left|\sum_{r=N}^{M} K_r(w,x,y)\right| &\leq \frac{1-x}{|x-y|}\left|\sum_{r=N}^{M} b_r p_r(\overline{w},x)p_r(w,y)\right| + \frac{1-y}{|x-y|}\left|\sum_{r=N}^{M} b_r p_r(\overline{w},y)p_r(w,x)\right| \\ &\leq \frac{\mathcal{C}}{|x-y|\sqrt{w(x)\varphi(x)}\sqrt{w(y)\varphi(y)}}\left[\Sigma_1\sqrt{1-x} + \Sigma_2\sqrt{1-y}\right] \qquad (49)\end{aligned}$$

where for $i = 1, 2$ we set

$$\Sigma_i := \left|\sum_{k=N}^{M} c_k \left(\cos(kt+\nu_i) + \frac{\mathcal{O}(1)}{k\sin t}\right)\left(\cos(ks+\mu_i) + \frac{\mathcal{O}(1)}{k\sin s}\right)\right|,$$

being ν_i, μ_i independent of k and

$$c_k = \frac{\sqrt{2}}{\pi}\frac{\Gamma(k)\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \sim 1.$$

From (49) we are going to deduce the statement (45) with the minus sign, by proving that

$$\left[\Sigma_1\sqrt{1-x} + \Sigma_2\sqrt{1-y}\right] \leq \mathcal{C}E_N^-(x,y). \qquad (50)$$

To this aim, we apply to the sums Σ_i the summation by part formula (20) with $a_j = c_j$, and taking into account that c_k is a bounded decreasing sequence w.r.t. k, we get

$$\Sigma_i \leq \mathcal{C}\left(\sup_{N\leq j\leq M}\left|\sum_{k=N}^{j}\left(\cos(kt+\nu_i) + \frac{\mathcal{O}(1)}{k\sin t}\right)\left(\cos(ks+\mu_i) + \frac{\mathcal{O}(1)}{k\sin s}\right)\right|\right), \qquad i = 1, 2.$$

Then, using well–known trigonometric identities and observing that for all $N \leq j \leq M$, we have

$$\left|\sum_{k=N}^{j}\cos(ka+b)\right| = \left|\sum_{k=N}^{j}\frac{\Delta\sin[(k-\frac{1}{2})a+b]}{2\sin\frac{a}{2}}\right| \leq \frac{\mathcal{C}}{|\sin\frac{a}{2}|}, \qquad (51)$$

$$\sum_{k=N}^{j}\frac{1}{k^2} \leq \sum_{k=N}^{M}\frac{1}{k^2} \leq \int_{N-1}^{M}\frac{dx}{x^2} = \frac{M-N+1}{M(N-1)} \leq \frac{\mathcal{C}}{N}, \qquad (52)$$

we obtain

$$\begin{aligned}\Sigma_i &\leq \frac{\mathcal{C}}{\sin\frac{s+t}{2}} + \frac{\mathcal{C}}{\sin\frac{|s-t|}{2}} + \frac{\mathcal{C}}{N}\left(\frac{1}{\sin t\sin\frac{s}{2}} + \frac{1}{\sin s\sin\frac{t}{2}} + \frac{1}{\sin t\sin s}\right) \\ &\leq \frac{\mathcal{C}}{|\sqrt{1-x}-\sqrt{1-y}|} + \frac{\mathcal{C}}{N\sqrt{1-x^2}\sqrt{1-y^2}} \\ &= \mathcal{C}\frac{\sqrt{1-x}+\sqrt{1-y}}{|x-y|} + \frac{\mathcal{C}}{N\sqrt{1-x^2}\sqrt{1-y^2}}, \qquad i = 1, 2,\end{aligned}$$

which yields (50) and consequently (45) with the minus sign.
Finally, the plus sign in (45) can be deduced by using the identity (cf. [15, (4.1.3)])

$$K_n(v^{\alpha,\beta}, x, y) = K_n(v^{\beta,\alpha}, -x, -y),$$

and applying to $\sum_{r=N}^{M} K_r(v^{\beta,\alpha}, -x, -y)$ the already stated equation (45) with the minus sign. $\diamondsuit$

By taking into account (44), the previous lemma yields the next pointwise estimate of the GVP kernel.

Lemma 5.2. *Let w be a given Jacobi weight and $N < M$ be positive integers. Moreover assume that the VP kernels $v^{N,M}(w,x,y)$ are defined by filter coefficients satisfying (10) and (28).*
Then, for all $x \neq y \in \left[-1+\frac{\mathcal{C}}{N^2}, 1-\frac{\mathcal{C}}{N^2}\right]$, we have

$$\left|v^{N,M}(w,x,y)\right| \leq \frac{\mathcal{C}}{N} \cdot \frac{E_N^{\pm}(x,y)}{|x-y|\sqrt{w(x)\varphi(x)}\sqrt{w(y)\varphi(y)}}, \qquad C \neq C(N,M,x,y), \tag{53}$$

being $\varphi(x) := \sqrt{1-x^2}$ and $E_N^{\pm}(x,y)$ given by (46).

Finally, using Lemma 5.2, we get the next two lemmas, which have been used in proving the theorems of the previous sections.

Lemma 5.3. *Assume that $w = v^{\alpha,\beta}$ and $u = v^{\gamma,\delta}$ satisfy (26) and the following conditions*

$$\begin{cases} \frac{\alpha}{2}-\frac{1}{4} < \gamma \leq \frac{\alpha}{2}+\frac{5}{4} & \text{and} \quad 0 \leq \gamma < \alpha+1, \\ \frac{\beta}{2}-\frac{1}{4} < \delta \leq \frac{\beta}{2}+\frac{5}{4} & \text{and} \quad 0 \leq \delta < \beta+1, \end{cases} \tag{54}$$

Moreover, let the GVP kernels $v^{N,M}(w,x,y)$ be defined by uniformly bounded filter coefficients $h_j^{N,M}$ satisfying (10) and (28). Then for all pairs of positive integers $N < M$ such that $N \sim M$, we have

$$\sup_{|x|\leq 1-\frac{\mathcal{C}}{M^2}} \left[u(x)\int_{-1}^{1} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy\right] \leq \mathcal{C}, \qquad \mathcal{C} \neq \mathcal{C}(N,M). \tag{55}$$

Proof. Since (54) assures that $wu^{-1} \in L^1$, we can apply Remez inequality (30) with $p=1$, $P(y) = v^{N,M}(w,x,y)$ and $v = wu^{-1}$, obtaining

$$u(x)\int_{-1}^{1} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy \leq \mathcal{C}u(x)\int_{-1+\frac{\mathcal{C}}{M^2}}^{1-\frac{\mathcal{C}}{M^2}} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy, \qquad \forall |x| \leq 1-\frac{\mathcal{C}}{M^2}.$$

Then, recalling that $N \sim M$, we are going to prove that

$$u(x)\int_{-1+\frac{\mathcal{C}}{N^2}}^{1-\frac{\mathcal{C}}{N^2}} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy \leq \mathcal{C}, \qquad \forall |x| \leq 1-\frac{\mathcal{C}}{N^2}.$$

To this aim, we consider the following decomposition

$$\begin{aligned} & u(x)\int_{-1+\frac{\mathcal{C}}{N^2}}^{1-\frac{\mathcal{C}}{N^2}} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy \\ = \quad & u(x)\left\{\int_{|y-x|\leq\frac{\sqrt{1-|x|}}{N}} + \int_{\frac{\sqrt{1-|x|}}{N}\leq|y-x|\leq\frac{1-|x|}{2}} + \int_{|y-x|\geq\frac{1-|x|}{2}}\right\} |v^{N,M}(w,x,y)|\frac{w(y)}{u(y)}dy \\ =: \quad & I_1(x)+I_2(x)+I_3(x) \end{aligned}$$

and estimate the addenda $I_k(x)$ by taking into account the next properties, which can be easily proved for any Jacobi weight $v^{\rho,\sigma}(x) = (1-x)^\rho(1+x)^\sigma$

$$\begin{aligned} (i) \quad & 1+x \geq \mathcal{C} \implies v^{\rho,\sigma}(x) \sim (1-x)^\rho \\ (ii) \quad & 1-x \geq \mathcal{C} \implies v^{\rho,\sigma}(x) \sim (1+x)^\sigma \\ (iii) \quad & |y-x| \leq \tfrac{1-|x|}{2} \implies v^{\rho,\sigma}(y) \sim v^{\rho,\sigma}(x) \end{aligned}$$

Estimate of $I_1(x)$. Suppose for instance that $x \geq 0$, the proof being similar when $x < 0$. Starting from (41), using $|h_j^{N,M}| \leq \mathcal{C} \neq \mathcal{C}(j)$, (iii), (i), and recalling that (see e.g. [9, p.138])

$$|p_j(w,x)| \leq \mathcal{C}\left(\sqrt{1-x}+\frac{1}{j}\right)^{-\alpha-\frac{1}{2}}\left(\sqrt{1+x}+\frac{1}{j}\right)^{-\beta-\frac{1}{2}}, \qquad |x| \leq 1, \qquad j \in \mathbb{N}, \qquad (56)$$

we have

$$\begin{aligned}
I_1(x) &\leq u(x)\int_{|y-x|\leq\frac{\sqrt{1-|x|}}{N}} \sum_{j=0}^{M} |h_j^{N,M}\ p_j(w,x)p_j(w,y)|\frac{w(y)}{u(y)}dy \\
&\leq \mathcal{C}w(x)\int_{|y-x|\leq\frac{\sqrt{1-|x|}}{N}} \sum_{j=0}^{M} |p_j(w,x)p_j(w,y)|dy \\
&\leq \mathcal{C}w(x)\left[1+\sum_{j=1}^{M}\left(\sqrt{1-x}+\frac{1}{j}\right)^{-2\alpha-1}\left(\sqrt{1+x}+\frac{1}{j}\right)^{-2\beta-1}\right]\int_{|y-x|\leq\frac{\sqrt{1-|x|}}{N}} dy \\
&\leq \frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\left[1+\sum_{j=1}^{M}\left(\sqrt{1-x}+\frac{1}{j}\right)^{-2\alpha-1}\right] \\
&\leq \mathcal{C}(1-x)^{\alpha+1}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\sum_{j=1}^{M}\left(\sqrt{1-x}+\frac{1}{j}\right)^{-2\alpha-1} \\
&\leq \mathcal{C}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\sum_{j=1}^{M}\left(\sqrt{1-x}+\frac{1}{j}\right)^{-2\alpha-1},
\end{aligned}$$

having used $\frac{\mathcal{C}}{N} \leq \sqrt{1-|x|}$ and $\alpha+1>0$ in the last two inequalities.
Hence, in the case $2\alpha+1 \geq 0$, we obtain

$$I_1(x) \leq \mathcal{C}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\sum_{j=1}^{M}(1-x)^{-\alpha-\frac{1}{2}} \leq \mathcal{C}+\mathcal{C}\frac{M}{N} \leq \mathcal{C},$$

while if $2\alpha+1<0$, by taking into account that $(a+b)^\mu \leq (a^\mu+b^\mu)$ holds for any $a,b>0$ and $\mu=-2\alpha-1 \in]0,1[$, we get

$$\begin{aligned}
I_1(x) &\leq \mathcal{C}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\sum_{j=1}^{M}\left[\sqrt{1-x}+\frac{1}{j}\right]^{-2\alpha-1} \\
&\leq \mathcal{C}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\sum_{j=1}^{M}\left[(1-x)^{-\alpha-\frac{1}{2}}+j^{2\alpha+1}\right] \\
&= \mathcal{C}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\left[M(1-x)^{-\alpha-\frac{1}{2}}+1+\sum_{j=2}^{M}j^{2\alpha+1}\right] \\
&\leq \mathcal{C}+\mathcal{C}\frac{M}{N}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}+\frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}}\int_1^M t^{2\alpha+1}dt \\
&\leq \mathcal{C}+\mathcal{C}(1-x)^{\alpha+1}+\mathcal{C}\left[M^2(1-x)\right]^{\alpha+\frac{1}{2}} \leq \mathcal{C},
\end{aligned}$$

having used $N \sim M$, $\frac{\mathcal{C}}{M^2} \leq 1-|x|$ and $\alpha>-1$ in the last inequalities.

Estimate of $I_2(x)$. We use (53) with plus or minus sign depending on whether x is negative or nonnegative respectively. In both the cases, by (i)–(iii) we observe that $E_N^{\pm}(x,y) \le \mathcal{C}\frac{1\pm x}{|y-x|}$ follows from $\frac{\sqrt{1-|x|}}{N} \le |y-x| \le \frac{1-|x|}{2}$. Consequently, by (53) and (iii), we get

$$\begin{aligned} I_2(x) &\le \frac{\mathcal{C}}{N}u(x)\int_{\frac{\sqrt{1-|x|}}{N}\le|y-x|\le\frac{1-|x|}{2}}\left[\frac{(1-|x|)}{|y-x|^2\sqrt{w\varphi(x)}\sqrt{w\varphi(y)}}\right]\frac{w(y)}{u(y)}dy \\ &\le \frac{\mathcal{C}}{N}\sqrt{1-|x|}\int_{\frac{\sqrt{1-|x|}}{N}\le|y-x|\le\frac{1-|x|}{2}}\left[\frac{dy}{|y-x|^2}\right] \le \mathcal{C}. \end{aligned}$$

Estimate of $I_3(x)$. As in the previous case, we use (53) with plus or minus sign depending on whether x is negative or nonnegative respectively, but now we note that $|y-x| \ge \frac{1-|x|}{2}$ yields $E_N^{\pm}(x,y) \le \mathcal{C}$. Hence by (53), we get

$$I_3(x) \le \frac{\mathcal{C}}{N}v^{\gamma-\frac{\alpha}{2}-\frac14,\ \delta-\frac{\beta}{2}-\frac14}(x)\int_{|y-x|\ge\frac{1-|x|}{2}}\frac{v^{-\gamma+\frac{\alpha}{2}-\frac14,\ -\delta+\frac{\beta}{2}-\frac14}(y)}{|y-x|}dy \tag{57}$$

Now suppose for instance that $x \ge 0$ (the case $x<0$ being analogous), and observe that by virtue of $(i),(ii)$, we have

$$\begin{aligned} I_3(x) &\le \frac{\mathcal{C}}{N}v^{\gamma-\frac{\alpha}{2}-\frac14,\delta-\frac{\beta}{2}-\frac14}(x)\left[\int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}+\int_{-\frac12}^{x-\frac{1-x}{2}}+\int_{x+\frac{1-x}{2}}^{1-\frac{\mathcal{C}}{N^2}}\right]\frac{v^{-\gamma+\frac{\alpha}{2}-\frac14,-\delta+\frac{\beta}{2}-\frac14}(y)}{|y-x|}dy \\ &\le \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14}\int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}(1+y)^{-\delta+\frac{\beta}{2}-\frac14}dy+ \\ &+ \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14}\int_{-\frac12}^{x-\frac{1-x}{2}}\frac{(1-y)^{-\gamma+\frac{\alpha}{2}-\frac14}}{x-y}dy \\ &+ \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14}\int_{x+\frac{1-x}{2}}^{1-\frac{\mathcal{C}}{N^2}}\frac{(1-y)^{-\gamma+\frac{\alpha}{2}-\frac14}}{y-x}dy =: J_1(x)+J_2(x)+J_3(x). \end{aligned}$$

In order to estimate $J_1(x)$, we consider the following cases:

Case: $\gamma \ge \frac{\alpha}{2}+\frac14$. Then we have $(1-x)^{\gamma-\frac{\alpha}{2}-\frac14} \le \mathcal{C}$ and consequently

$$\begin{aligned} J_1(x) &:= \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14}\int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}(1+y)^{-\delta+\frac{\beta}{2}-\frac14}dy \le \frac{\mathcal{C}}{N}\int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}(1+y)^{-\delta+\frac{\beta}{2}-\frac14}dy \\ &\le \begin{cases} \frac{\mathcal{C}}{N}\int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}(1+y)^{-\frac32}dy \le \mathcal{C} & \text{if } \delta = \frac{\beta}{2}+\frac54, \\ \int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12}(1+y)^{-\delta+\frac{\beta}{2}+\frac14}dy \le \mathcal{C} & \text{if } \delta < \frac{\beta}{2}+\frac54. \end{cases} \end{aligned}$$

Case: $\frac{\alpha}{2}-\frac14 < \gamma < \frac{\alpha}{2}+\frac14$. In this case we observe that

$$\begin{aligned} 1-x \ge \frac{\mathcal{C}}{N^2} &\implies (1-x)^{\gamma-\frac{\alpha}{2}-\frac14} \le \left(\frac{\mathcal{C}}{N^2}\right)^{\gamma-\frac{\alpha}{2}-\frac14}, \\ 1+y \ge \frac{\mathcal{C}}{N^2} &\implies (1+y)^{\gamma-\frac{\alpha}{2}+\frac14} \ge \left(\frac{\mathcal{C}}{N^2}\right)^{\gamma-\frac{\alpha}{2}+\frac14}, \end{aligned}$$

and consequently we get

$$J_1(x) \le \left(\frac{\mathcal{C}}{N^2}\right)^{\gamma-\frac{\alpha}{2}+\frac14} \int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12} (1+y)^{-\delta+\frac{\beta}{2}-\frac14} dy \le \mathcal{C} \int_{-1+\frac{\mathcal{C}}{N^2}}^{-\frac12} (1+y)^{-\delta+\frac{\beta}{2}+\gamma-\frac{\alpha}{2}} dy \le \mathcal{C},$$

having used (26) in the last inequality.

As regards J_2, by applying the substitution $t = \frac{1-y}{1-x}$, we get

$$J_2(x) := \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14} \int_{-\frac12}^{x-\frac{1-x}{2}} \frac{(1-y)^{-\gamma+\frac{\alpha}{2}-\frac14}(y)}{x-y} dy = \frac{\mathcal{C}}{N\sqrt{1-x}} \int_{\frac32}^{\frac{3}{2(1-x)}} \frac{t^{-\gamma+\frac{\alpha}{2}-\frac14}}{t-1} dt.$$

Then taking into account that $t > \frac32 \Longrightarrow t-1 \ge \frac{t}{3}$, recalling that $\frac{\mathcal{C}}{N\sqrt{1-x}} \le \mathcal{C}$ and using the assumption $\gamma > \frac{\alpha}{2} - \frac14$, we conclude

$$J_2(x) \le \mathcal{C} \int_{\frac32}^{\infty} t^{-\gamma+\frac{\alpha}{2}-\frac54} dt \le \mathcal{C}.$$

Finally, concerning J_3, we use the other hypothesis $\gamma \le \frac{\alpha}{2} + \frac54$. In the limiting case $\gamma = \frac{\alpha}{2} + \frac54$, using $(1-x) \le 2(y-x)$, we have

$$J_3(x) = \frac{\mathcal{C}}{N}(1-x) \int_{x+\frac{1-x}{2}}^{1-\frac{\mathcal{C}}{N^2}} \frac{(1-y)^{-\frac32}(y)}{y-x} dy \le \frac{\mathcal{C}}{N} \int_{x+\frac{1-x}{2}}^{1-\frac{\mathcal{C}}{N^2}} (1-y)^{-\frac32}(y) dy \le \mathcal{C},$$

while if $\gamma < \frac{\alpha}{2} + \frac54$, taking into account that $\frac{\mathcal{C}}{N} \le \sqrt{1-y}$, and substituting $t = \frac{1-y}{1-x}$, we get

$$\begin{aligned} J_3(x) &\le \mathcal{C}(1-x)^{\gamma-\frac{\alpha}{2}-\frac14} \int_{x+\frac{1-x}{2}}^{1-\frac{\mathcal{C}}{N^2}} (1-y)^{-\gamma+\frac{\alpha}{2}+\frac14}(y) dy \\ &\le \mathcal{C} \int_0^{\frac12} \frac{t^{-\gamma+\frac{\alpha}{2}+\frac14}}{1-t} dt \le \mathcal{C} \int_0^{\frac12} t^{-\gamma+\frac{\alpha}{2}+\frac14} dt \le \mathcal{C}. \end{aligned}$$

$\diamondsuit$

Lemma 5.4. *Assume that $w = v^{\alpha,\beta}$ $(\alpha, \beta > -1)$ and $u = v^{\gamma,\delta}$ $(\gamma, \delta \ge 0)$ satisfy (34)–(36). Moreover, let the GVP kernels $v^{N,M}(w, x, y)$ be defined by filter coefficients $h_j^{N,M}$ satisfying (10) and (28). Then for all positive integers $n \sim N \sim M$, such that $N < M$, we have*

$$\sup_{|x| \le 1-\frac{\mathcal{C}}{M^2}} \left[u(x) \sum_{k=1}^{n} |v^{N,M}(w, x, x_k)| \frac{w(x_k)}{u(x_k)} \Delta x_k \right] \le \mathcal{C}, \qquad \mathcal{C} \ne \mathcal{C}(N, M), \tag{58}$$

where x_k, $k = 1, \dots, n$, are the zeros of $p_n(w, x)$ given in (38), and $\Delta x_k = x_{k+1} - x_k$.

Proof. The statement can be achieved analogously to the previous lemma. More precisely, for any fixed $|x| \le 1 - \frac{\mathcal{C}}{M^2}$, we consider the following indices sets

$$\begin{aligned} \mathcal{K}_1 &:= \left\{ k \ : \ |x - x_k| \le \frac{\sqrt{1-|x|}}{N} \right\}, \\ \mathcal{K}_2 &:= \left\{ k \ : \ \frac{\sqrt{1-|x|}}{N} \le |x - x_k| \le \frac{1-|x|}{2} \right\}, \\ \mathcal{K}_3 &:= \left\{ k \ : \ |x - x_k| \ge \frac{1-|x|}{2} \right\}, \end{aligned}$$

and we are going to estimate separately the following sums

$$s_i(x) := u(x) \sum_{k \in \mathcal{K}_i} |v^{N,M}(w,x,x_k)| \frac{w(x_k)}{u(x_k)} \Delta x_k, \qquad i = 1,2,3,$$

taking into account that [12]

$$\Delta x_{k\pm 1} \sim \Delta x_k \sim \frac{\sqrt{1-|t|}}{n}, \qquad x_k \le t \le x_{k+1}. \tag{59}$$

As regards $s_1(x)$, similarly to the estimate of $I_1(x)$ in the previous proof, by means of (41), (56) and (i)–(iii), supposed for instance that $x \ge 0$, we get

$$s_1(x) \le \mathcal{C}(1-x)^\alpha \sum_{j=1}^{M} \left(\sqrt{1-x} + \frac{1}{j}\right)^{-2\alpha-1} \sum_{k \in \mathcal{K}_1} \Delta x_k.$$

On the other hand, from (59) and the definition of $\mathcal{K}_1$ we easily deduce

$$\sum_{k \in \mathcal{K}_1} \Delta x_k \le \mathcal{C} \frac{\sqrt{1-|x|}}{N},$$

hence, following the same reasoning of the previous proof, we conclude

$$s_1(x) \le \frac{\mathcal{C}}{N}(1-x)^{\alpha+\frac{1}{2}} \sum_{j=1}^{M} \left(\sqrt{1-x} + \frac{1}{j}\right)^{-2\alpha-1} \le \mathcal{C}.$$

Concerning $s_2(x)$, similarly to the estimate of $I_2(x)$ in the previous proof, using (53) and (59), we get

$$s_2(x) \le \frac{\mathcal{C}}{N}\sqrt{1-|x|} \sum_{k \in \mathcal{K}_2} \frac{\Delta x_k}{|x-x_k|^2} \le \frac{\mathcal{C}}{N}\sqrt{1-|x|} \int_{\frac{\sqrt{1-|x|}}{N} \le |t-x| \le \frac{1-|x|}{2}} \frac{dt}{(t-x)^2} \le \mathcal{C}.$$

Finally, as regards $s_3(x)$, similarly to $I_3(x)$, by (53) we get

$$s_3(x) \le \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{\delta-\frac{\beta}{2}-\frac{1}{4}} \sum_{k \in \mathcal{K}_3} \frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}-\frac{1}{4}}(1+x_k)^{-\delta+\frac{\beta}{2}-\frac{1}{4}}}{|x-x_k|} \Delta x_k \tag{60}$$

Hence, if for instance we assume $x \ge 0$ (the case $x < 0$ being analogous), we get

$$\begin{aligned}
s_3(x) &\le \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \sum_{x_1 \le x_k \le -\frac{1}{2}} (1+x_k)^{-\delta+\frac{\beta}{2}-\frac{1}{4}} \Delta x_k \\
&+ \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \left[\sum_{-\frac{1}{2} \le x_k \le x - \frac{1-x}{2}} + \sum_{x + \frac{1-x}{2} \le x_k \le x_n}\right] \frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}-\frac{1}{4}}}{|x-x_k|} \Delta x_k \\
&\le \frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \left[\int_{-1+\frac{\mathcal{C}}{N^2}}^{0} (1+y)^{-\delta+\frac{\beta}{2}-\frac{1}{4}} dy + \sum_{\substack{k=1 \\ k \ne d}} \frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}-\frac{1}{4}}}{|x-x_k|} \Delta x_k\right]
\end{aligned}$$

where d denotes the index of the node closest to x.
As regards the first addendum, by following the same reasoning of the previous proof in estimating $J_1(x)$, we get

$$\frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\int_{-1+\frac{\mathcal{C}}{N^2}}^{0}(1+y)^{-\delta+\frac{\beta}{2}-\frac{1}{4}}dy\leq\mathcal{C}.$$

Finally, we estimate the second addendum by taking into account that [9]

$$\sum_{\substack{k=1\\k\neq d}}\frac{(1\pm x_k)^{\rho}}{|x-x_k|}\Delta x_k\leq\mathcal{C}(1\pm x)^{\rho},\qquad -1\leq\rho<0. \tag{61}$$

More precisely, if $\frac{\alpha}{2}+\frac{3}{4}<\gamma\leq\frac{\alpha}{2}+\frac{5}{4}$ then we use (61) with $\rho=-\gamma+\frac{\alpha}{2}-\frac{1}{4}$ and we have

$$\frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\sum_{\substack{k=1\\k\neq d}}\frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}-\frac{1}{4}}}{|x-x_k|}\Delta x_k\leq\frac{\mathcal{C}}{N\sqrt{1-x}}\leq\mathcal{C},$$

while in the case $\frac{\alpha}{2}-\frac{1}{4}\leq\gamma\leq\frac{\alpha}{2}+\frac{3}{4}$, we use $N\geq\mathcal{C}n\geq(1-x_k)^{-\frac{1}{2}}$ and applying (61) with $\rho=-\gamma+\frac{\alpha}{2}+\frac{1}{4}$ we get

$$\frac{\mathcal{C}}{N}(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\sum_{\substack{k=1\\k\neq d}}\frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}-\frac{1}{4}}}{|x-x_k|}\Delta x_k\leq(1-x)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\sum_{\substack{k=1\\k\neq d}}\frac{(1-x_k)^{-\gamma+\frac{\alpha}{2}+\frac{1}{4}}}{|x-x_k|}\Delta x_k\leq\mathcal{C}.$$

♢

6. Numerical experiments

In this section, the results of several numerical experiments will be shown. To ensure that $N\sim M$, the values of N and M are coupled such that the parameter

$$\theta\approx\frac{M-N}{M+N}$$

is kept constant when N changes. More precisely, for a given N and $\theta\in[0,1)$, M is chosen as

$$M=\left\lfloor\frac{1+\theta}{1-\theta}N\right\rfloor. \tag{62}$$

6.1. Number of quadrature nodes

In this subsection, we investigate what a good choice would be for the parameter n, i.e., the number of quadrature nodes when computing a discrete GVP mean (17). In contrast to the continuous GVP mean, we have this number n as an additional parameter besides the degrees N and M. In the literature (see e.g. [16]), n is chosen as the smallest integer n_* such that $2n_*>M+N$. We will see that this choice is not the best, especially for small values of the parameter θ.

We take the following values for the parameters: $\alpha = \beta = \gamma = \delta = 0$, $N = 50$, $\theta = 0.1, 0.2, \ldots, 0.9$ and M given by (62). Moreover, as filter coefficients, we choose the classical VP filter coefficients linearly decreasing from 1 to 0 between N and M, namely

$$h(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 2-x & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}, \qquad h_j^{N,M} = h\left(1 + \frac{j-N}{M+1}\right), \quad j = 0, 1, \ldots \tag{63}$$

Let us compute the Lebesgue constant (LC) for $\tilde{V}_n^{N,M}$ with $n = n_*, n_*+1, \ldots, 2M$. The result is shown in Figure 1 where the horizontal axis is $n/(M+1)$. The red circles indicate the LC for $n/(M+1) = 1$. This figure indicates that a good choice for n is $M+1$.

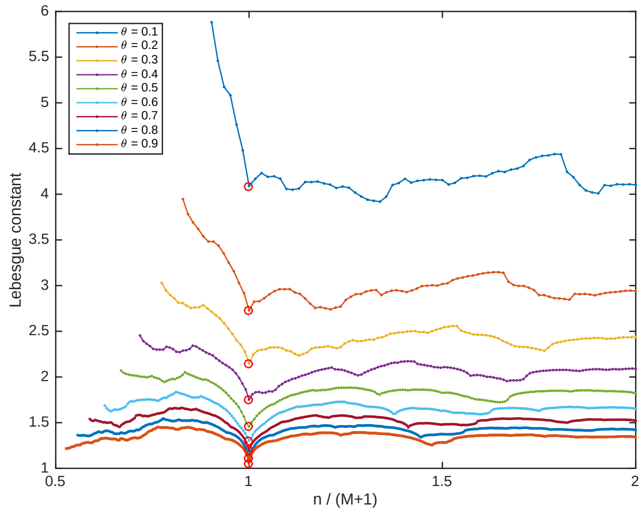

Figure 1: Lebesgue constants of $\tilde{V}_n^{N,M}$ for $N = 50$, $\theta = 0.1, 0.2, \ldots, 0.9$ and $n = n_*, n_*+1, \ldots, 2M$ with $\alpha = \beta = \gamma = \delta = 0$.

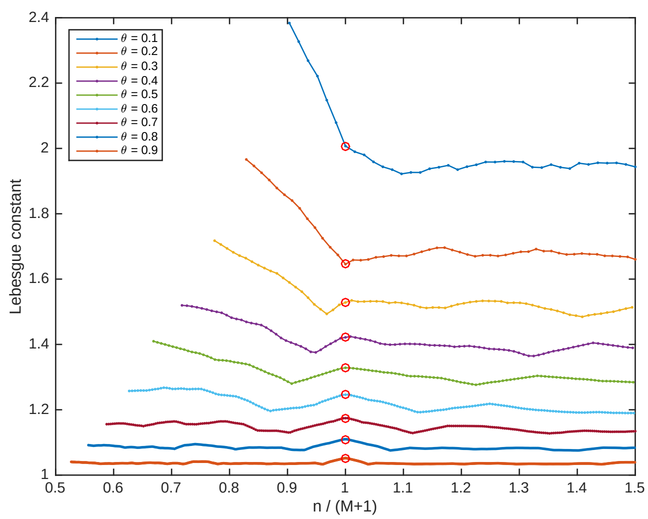

Figure 2: Lebesgue constant of $\tilde{V}_n^{N,M}$ for $N = 50$ and $n = n_*, n_*+1, \ldots, 1.5M$ with $\alpha = \beta = 0.5$ and $\gamma = \delta = 0$.

Figure 2 displays the results obtained for different values of α and β, which now we take equal to $1/2$. In this case $n = M+1$ is a reasonable choice especially for small values of

θ, but for larger values of θ a smaller value of n would result in a smaller LC. However, looking at the scale of the vertical axis, the improvement is negligible.

We have observed the same behavior also for other values of the parameters α, β, γ and δ. Hence, from now on, we will take n equal to $M+1$.

6.2. *Lebesgue constants in function of θ*

The results of the previous subsection show that for a fixed value of N increasing the value of θ, i.e., increasing the value of M, decreases the LC for the discrete GVP mean. Let us illustrate this for the parameters: $\alpha=\beta=\gamma=\delta=0$, $p=\infty$, $N=[50,100,200,400]$, $\theta=0.05,0.1,0.15,\dots,0.9$. Let us compute the LC for $\tilde{V}_n^{N,M}$ with $n=M+1$ and with the filter function (63). The result is shown in Figure 3.

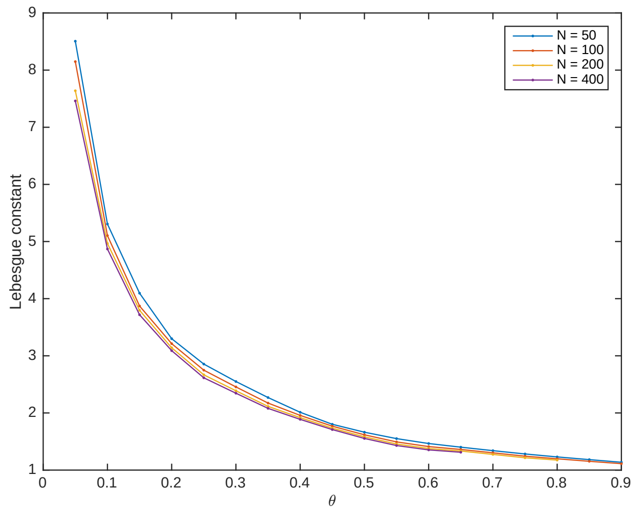

Figure 3: Lebesgue constant of $\tilde{V}_n^{N,M}$ for $N=[50,100,200,400]$, $\theta=0.05,0.1,0.15,\dots,0.9$ and $n=M+1$ with $\alpha=\beta=\gamma=\delta=0$.

6.3. *Uniform boundedness of Lebesgue constant for different examples*

In this subsection, we will take increasing values of N and study the uniform boundedness of the LC of both the continuous and the discrete GVP mean operator $V^{N,M}$ and $\tilde{V}_n^{N,M}$, respectively, for $p=\infty$ and $u=v^{\gamma,\delta}=1$. We will consider different filter coefficients and different values of $\theta=0.1,0.2,\dots,0.9$. We fix $u=v^{\gamma,\delta}=1$ and in the next examples we make two different choices of $w=v^{\alpha,\beta}$ such that the assumptions of both the theorems 3.1 and 4.1 are satisfied in the first example, but not in the second one.

Example 1: In this example the filter coefficients are the classical ones given by (63), and the weights' parameters for $w=v^{\alpha,\beta}$ and $u=v^{\gamma,\delta}$ are:

α	β	γ	δ
-0.5	-0.5	0	0

For such a choice the assumptions of both the Theorems 3.1 and 4.1 are satisfied, which ensure the uniform boundedness of the Lebesgue constants. This is confirmed by Figure 4, which shows the Lebesgue constants of the discrete and continuous GVP mean for the different values of $\theta=0.1,0.2,\dots,0.9$. Figure 5 shows the same plots but with the results added for $\theta=0$.

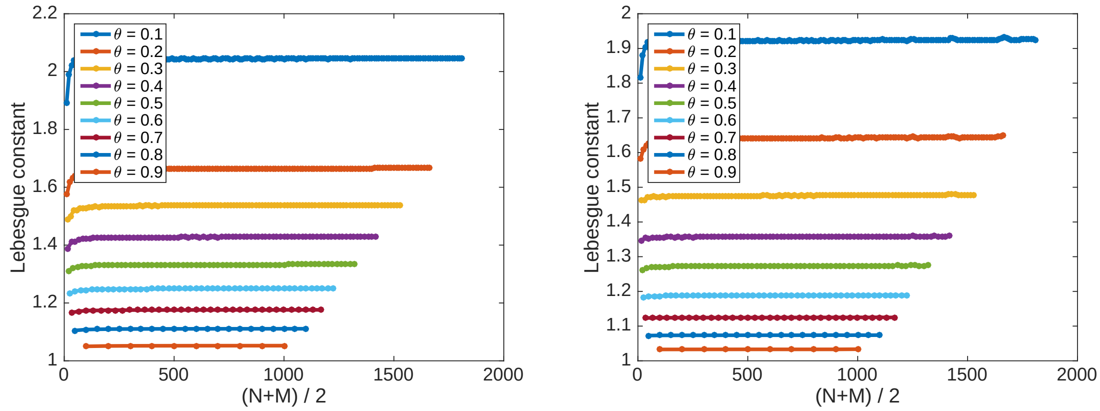

Figure 4: Lebesgue constant of discrete (left) and continuous (right) GVP mean for values of $\theta = 0.1, 0.2, \ldots, 0.9$ with $\alpha = -0.5$, $\beta = -0.5$, $\gamma = 0$ and $\delta = 0$.

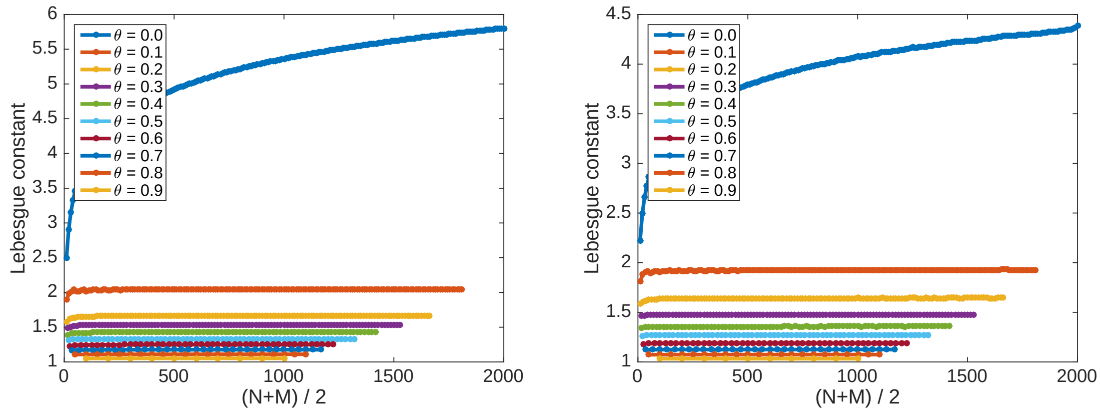

Figure 5: Lebesgue constant of discrete (left) and continuous (right) GVP mean for values of $\theta = 0.0, 0.1, 0.2, \ldots, 0.9$ with $\alpha = -0.5$, $\beta = -0.5$, $\gamma = 0$ and $\delta = 0$.

Note that $\theta = 0$ means $M = N$ and we get the Lagrange and Fourier projections (19) and (18) respectively whose Lebesgue constants are known to be unbounded (see e.g. [9]) growing with N as $\log N$. Indeed, from a computational point of view this is not very significative, nevertheless there is another nice feature of discrete VP approximation versus Lagrange interpolation: GVP means can be used to decrease the Gibbs phenomenon. This is illustrated in the next figure where we approximate the sign function by a polynomial of degree $M = 101$ for the values of θ equal to 0, 0.5 and 0.8 with $\alpha = -0.5$, $\beta = -0.5$, $\gamma = 0$ and $\delta = 0$. Figure 6 shows on the left–hand side the different discrete GVP mean polynomial approximants for the classical linear filter on the whole interval $[-1, 1]$ while on the right–hand side we zoom in on the discontinuity. For more details on resolving the Gibbs phenomenon, we refer the interested reader to [6].

Example 2: In this case we take the following weights' parameters, which do not satisfy Theorem 3.1 neither Theorem 4.1

α	β	γ	δ
1	0	0	0

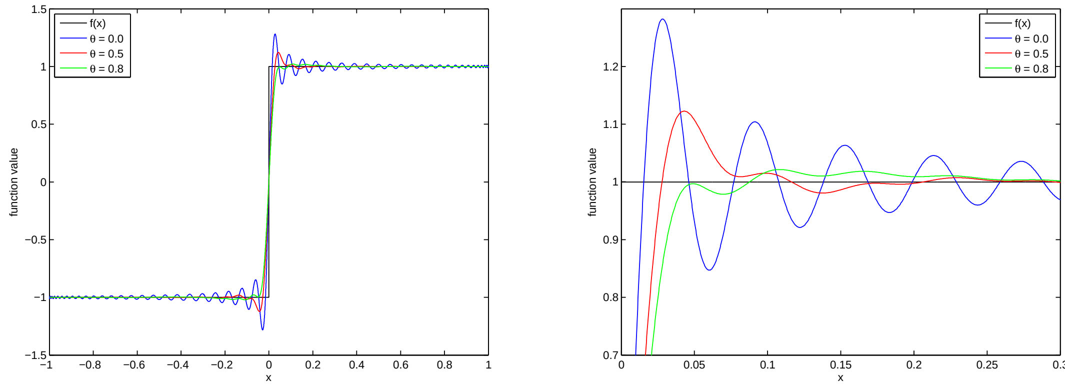

Figure 6: Discrete GVP mean approximants for the sign function for values of $\theta = 0.0, 0.5$ and 0.8, with $\alpha = -0.5$, $\beta = -0.5$, $\gamma = 0$ and $\delta = 0$.

Differently from the previous examples where we used the classical VP filter function (63), here we compare it with other nine different filter coefficients that have been used during the years (see [6, 2, 14] and the references therein). More precisely, from here on, we concern the following cases

Filter 1 Classical VP linear filter coefficients given by (63).

Filter 2 Lanczos filter coefficients given by (11) where the function h is null outside of $[0, 2]$, for all $x \in [0, 1]$ we have $h(x) = 1$, and

$$h(x) = \frac{\sin \pi(x-1)}{\pi(x-1)}, \qquad x \in (1, 2].$$

Filter 3 The same as before, but with the following second-order filter

$$h(x) = \frac{1 + \cos \pi(x-1)}{2}, \qquad x \in (1, 2].$$

Filter 4 The same as before, but for $x \in (1, 2]$ we take the following eighth–order filter defined in terms of $\sigma(\eta) = (1 + cos(\pi\eta))/2$ as follows

$$h(x) = \sigma^4(\eta) \left[35 - 84\sigma(\eta) + 70\sigma^2(\eta) - 20\sigma^3(\eta)\right], \qquad \eta = x - 1.$$

Filter 5 The same as in case 2, but for $x \in (1, 2]$ we take the following exponential filter of order p

$$h(x) = e^{-\alpha(x-1)^p}, \qquad e^{-\alpha} = \epsilon_{\text{mach}}.$$

Filter 6 The piecewise quadratic filter with support on $[0, a]$, with $a > 1$, $h(x) = 1$ for all $x \in [0, 1]$ and

$$h(x) = \begin{cases} 1 - 2y^2(x) & x \in [1, \frac{1+a}{2}] \\ 2(1 - y(x))^2 & x \in [\frac{1+a}{2}, a] \end{cases} \qquad y(x) = \frac{x-1}{a-1}.$$

Filter 7 The same as before, but for $x \in [1, a]$ we take the following cubic spline filter

$$h(x) = 1 - 3y^2(x) + 2y^3(x), \qquad y(x) = \frac{x-1}{a-1}, \qquad x \in [1, a].$$

Filter 8 The same as before, but for $x \in (1, a]$ we take the following C^∞ exponential filter

$$h(x) = \exp\left(\frac{-2e^{-\frac{2}{y(x)}}}{1 - y(x)}\right), \qquad y(x) = \frac{x-1}{a-1}.$$

Filter 9 The same as before, but for $x \in (1, a]$ we take the following C^∞ exponential filter having a reverse decay w.r.t. the previous case

$$h(x) = \exp\left(\frac{-2e^{-\frac{2}{z(x)}}}{1 - z(x)}\right), \qquad z(x) = \frac{a-x}{a-1}.$$

Filter 10 In this case the filter coefficients are those in formula (5).

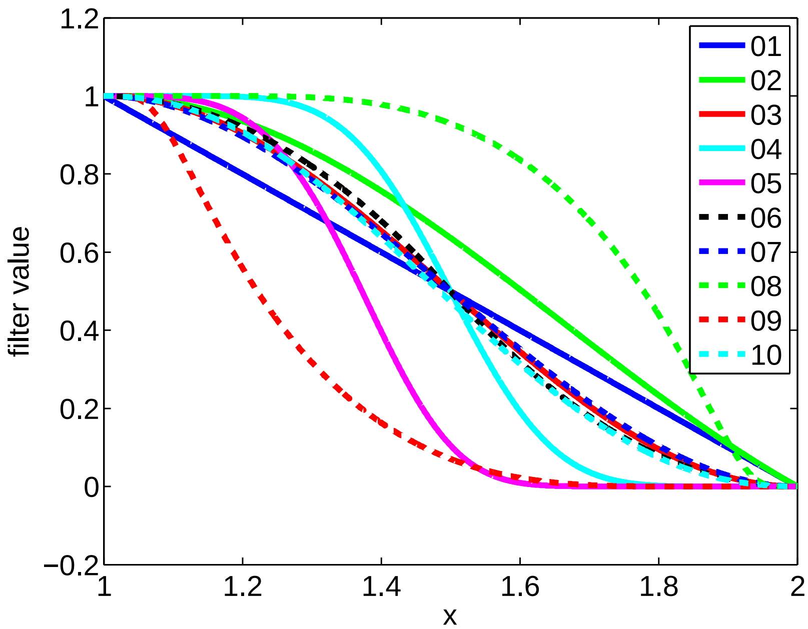

Figure 7: Plots of the different filter coefficients $h_j^{N,M}$, $j = N, \ldots, M+1$ from Example 2 (with $a = 2$ for filters 6–9).

In the sequel, we show the Lebesgue constants of the previous 10 filters (with $a = 2$ for filters 6–9) for different values of $\theta = 0.1, 0.2, \ldots, 0.9$ and both the discrete and continuous GVP mean. Even if the discrete GVP mean gives smaller Lebesgue constants compared to the continuous mean for the chosen parameter values, they are both not bounded for the classical VP filter (filter 1). However, filter 2 reveals uniformly bounded LC for the discrete GVP mean but not for the continuous one. The other filter choices reveal us uniform bounded LC for the discrete as well as for the continuous GVP mean. In Figure 11 the LC of all the previous discrete and continuous cases have been compared only for the smallest $\theta = 0.1$.

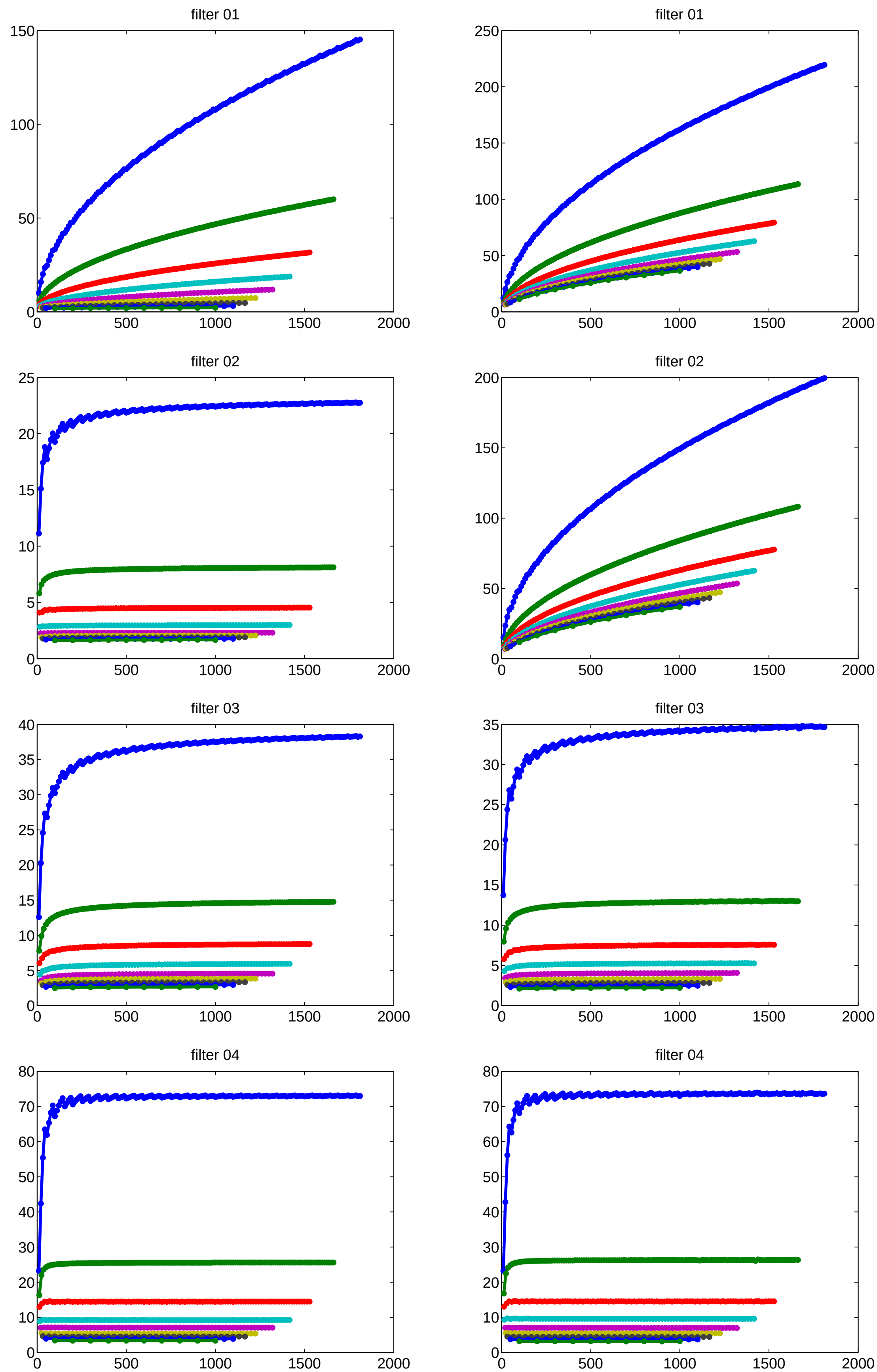

Figure 8: Lebesgue constant using filters 1 (top), 2, 3, and 4 (bottom) of discrete (left) and continuous (right) GVP mean in function of $(N+M)/2$ for values of $\theta = 0.1$ (top-line), $0.2, 0.3, \ldots, 0.9$ (bottom-line) with $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$.

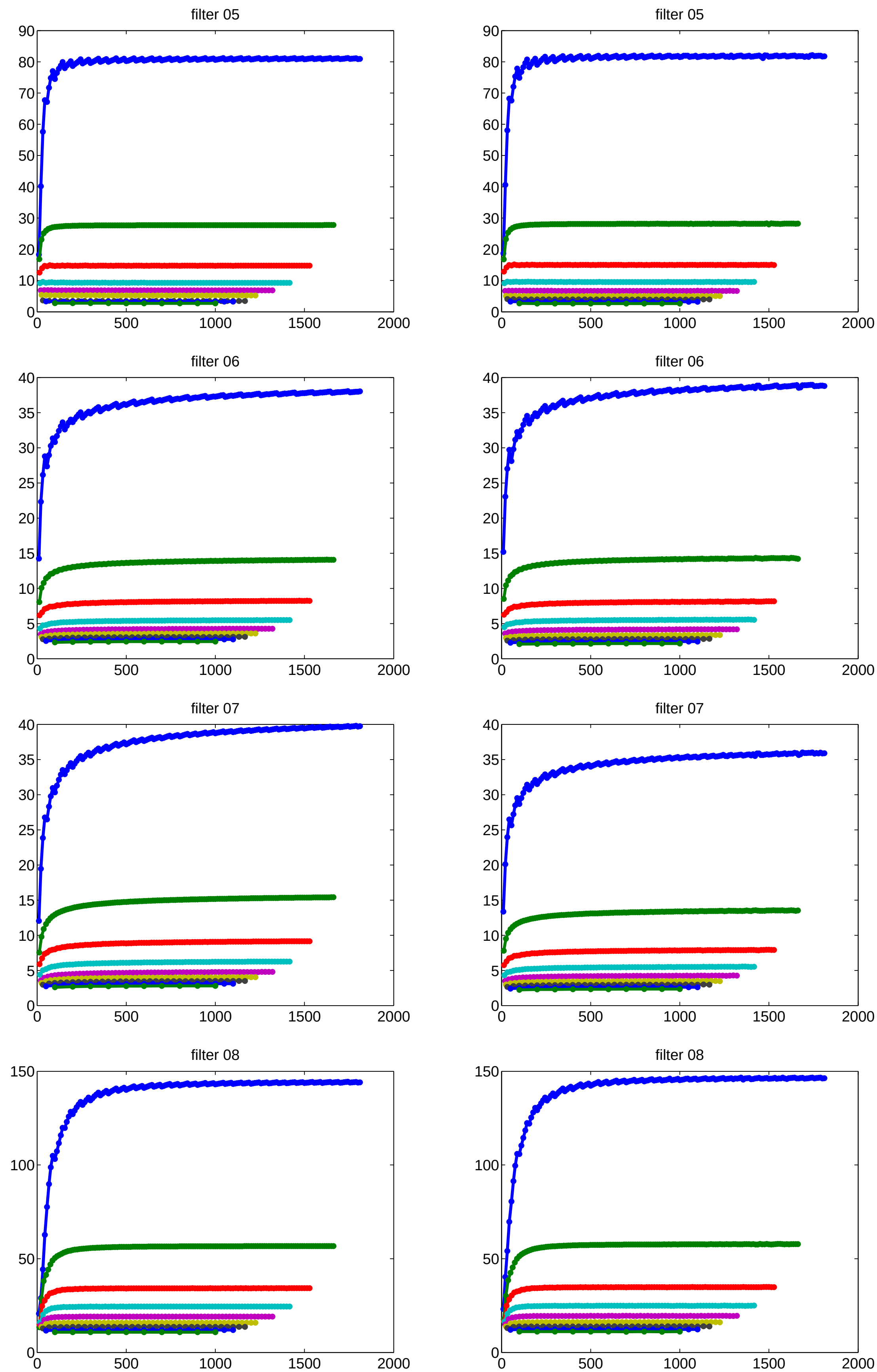

Figure 9: Lebesgue constant using filters 5 (top), 6, 7, and 8 (bottom) of discrete (left) and continuous (right) GVP mean in function of $(N+M)/2$ for values of $\theta = 0.1$ (top-line), $0.2, 0.3, \dots, 0.9$ (bottom-line) with $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$.

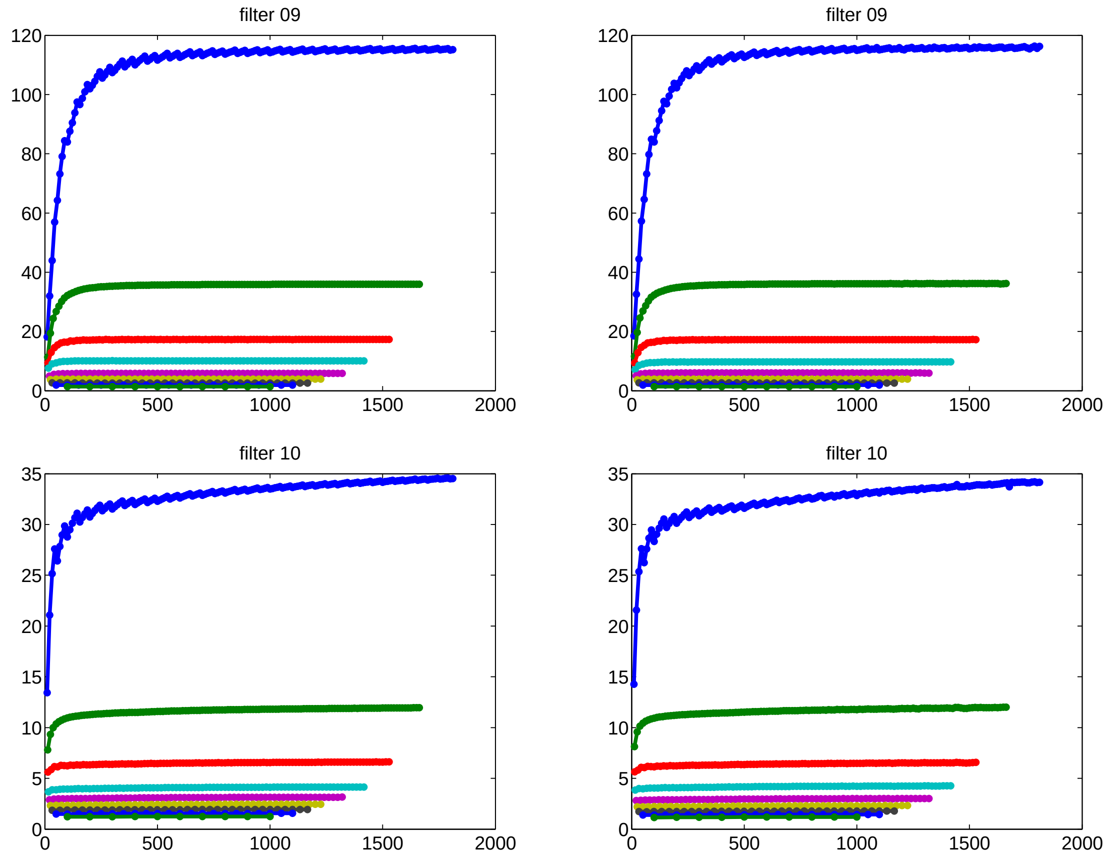

Figure 10: Lebesgue constant using filters 9 (top) and 10 (bottom) of discrete (left) and continuous (right) GVP mean in function of $(N+M)/2$ for values of $\theta = 0.1$ (top-line), $0.2, 0.3, \ldots, 0.9$ (bottom-line) with $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$.

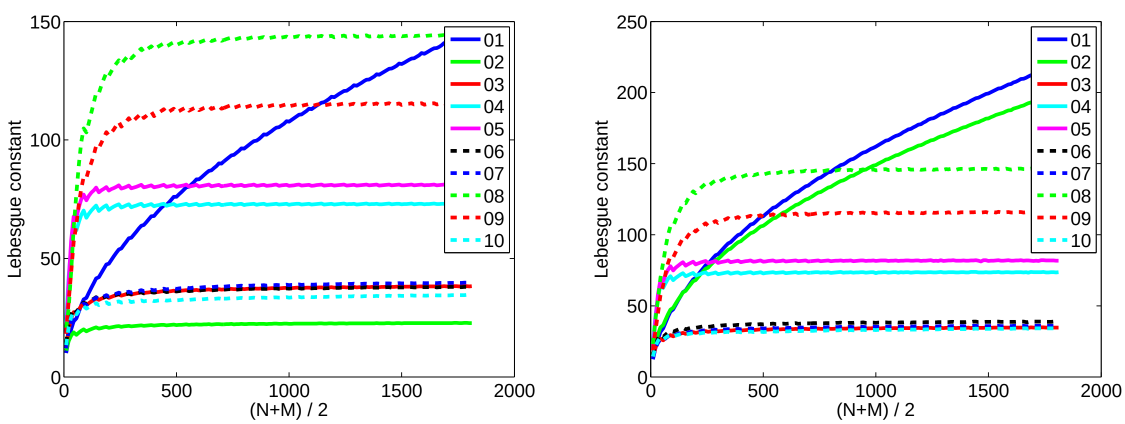

Figure 11: Lebesgue constant using filters 1 to 10 of discrete (left) and continuous (right) GVP mean for value of $\theta = 0.1$ with $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$.

7. Conclusion

In this paper, the continuous and discrete weighted approximation provided by the GVP quasi-projections (21) is studied. For both the continuous and the discrete GVP mean associated to a Jacobi weight $w = v^{\alpha,\beta}$, it is shown that the Lebesgue constants weighted by $u = v^{\gamma,\delta}$ are uniformly bounded when the parameters α, β, γ and δ satisfy the assumptions of Theorem 4.1 and Theorem 3.1, respectively. This result is true for any pair of mean parameters $N \sim M$ and any choice of the weight coefficients satisfying the decay requirement (8).

We have illustrated the theory by some examples, which reveal us that even if the theoretical bounds are not satisfied for the parameters α, β, γ and δ, there exist different choices of the filter which equally lead to uniformly bounded Lebesgue constants. Hence, under the assumption (8), the bounds stated for α, β, γ and δ are sufficient but not necessary conditions. This leaves open the problem of stating larger theoretical bounds, which also take into account the nature of the filter, instead of limiting us to assume (8).

Finally, comparing the discrete GVP approximation versus classical Lagrange interpolation at Jacobi abscissas, we have also illustrated that the GVP mean decreases the Gibbs phenomenon.

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